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CITATION:

三宅, 啓道. ON COMPACTNESS IN  $L^1$ . 数理解析研究所講究録 2015, 1963: 223-230: KJ00010016408.

ISSUE DATE:

2015-10

URL:

<http://hdl.handle.net/2433/224164>

RIGHT:

## ON COMPACTNESS IN $L^1$

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### 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, \mu)$  be a positive measure space with  $\sigma$ -algebra  $\mathcal{A}$  and measure  $\mu$  and let  $\mathcal{F}$  be the family of measurable subsets of  $\Omega$  with finite measure. Let  $L^1$  and  $L^\infty$  be the space of integrable functions defined on  $\Omega$  and the space of essentially-bounded measurable functions defined on  $\Omega$ , respectively. We denote by  $L_{loc}^\infty$  the vector subspace of  $L^\infty$  consisting of essentially-bounded measurable functions  $f$  defined on  $\Omega$  for which  $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$ . In [5], we discussed a method of constructing a separated locally convex topology  $\tilde{\tau}$  on  $L^1$  generated by the semi-norms  $f \mapsto \int_E |f| d\mu$  ( $E \in \mathcal{F}$ ) with the assumption that  $\mu$  is  $\sigma$ -finite. The topological dual of  $(L^1, \tilde{\tau})$  is algebraically isomorphic to  $L_{loc}^\infty$ . A notion of local uniform integrability for subsets of  $L^1$  was also discussed to obtain a necessary and sufficient condition for a bounded subset of  $L^1$  relative to  $L^1$ -norm to be relatively weakly compact in  $(L^1, \tilde{\tau})$ : Whenever  $C$  is a bounded subset of  $L^1$  relative to  $L^1$ -norm,  $C$  is locally uniformly integrable if and only if  $C$  is relatively weakly compact in  $(L^1, \tilde{\tau})$ . We applied it to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on  $L^1$ . This result gives an identification of the limit function in almost everywhere convergence of the Cesàro means  $n^{-1} \sum_{k=0}^{n-1} T^k f$  of an  $f \in L^1$ ; see [6] for details.

In this paper, we summarize the arguments presented in [7] and [8] about a characterization of compactness for the weak topology of  $L^1$  associated with  $\tau$ , and then apply similar arguments to discuss some necessary and sufficient conditions of compactness for the topology on  $L^1$  generated by the metric  $(f, g) \mapsto \int_\Omega |f - g| d\mu$  and the weak topology  $\sigma(L^1, L^\infty)$  on  $L^1$  generated by  $L^\infty$ , respectively. As their applications, (weak) almost periodicity of linear and non-linear operators in  $L^1$  is also discussed.

### 2. PRELIMINARIES

Throughout the paper, let  $\mathbb{N}_+$  and  $\mathbb{R}$  denote the set of non-negative integers and the set of real numbers, respectively. Let  $\langle E, F \rangle$  be a duality between vector spaces  $E$  and  $F$  over  $\mathbb{R}$ . If  $A$  is a subset of  $E$ , then  $A^\circ = \{y \in F : \langle x, y \rangle \leq 1 (x \in A)\}$  is a subset of  $F$ , called the polar of  $A$ . For each  $y \in F$ , we define a linear form  $f_y$  on  $E$  by

$f_y(x) = \langle x, y \rangle$  ( $x \in E$ ). Then,  $\sigma(E, F)$  denotes the weak topology on  $E$  generated by the family  $\{f_y : y \in F\}$  and  $\tau(E, F)$  denotes the Mackey topology on  $E$  with respect to  $\langle E, F \rangle$ , that is, the topology of uniform convergence on the circled, convex,  $\sigma(F, E)$ -compact subsets of  $F$ . Let  $(E, \mathfrak{T})$  is a locally convex space. Then, the topological dual of  $E$  is denoted by  $E'$ . The bilinear form  $(x, f) \mapsto f(x)$  on  $E \times E'$  defines a duality  $\langle E, E' \rangle$  and the weak topology on  $E$  generated by  $E'$  is called the weak topology of  $E$  (associated with  $\mathfrak{T}$  if this distinction is necessary). If  $E$  is a Banach space, then the subset  $\{x \in E : \|x\| \leq r\}$  of  $E$  is called the closed ball with center at 0 and radius  $r$ , denoted by  $B(r)$ . In particular,  $B(1)$  is called the closed unit ball in  $E$ .

Throughout the paper, let  $(\Omega, \mathcal{A}, \mu)$  denote a positive measure space with  $\sigma$ -algebra  $\mathcal{A}$  and measure  $\mu$ , and let  $\mathcal{F}$  denote the family of measurable subsets of  $\Omega$  with finite measure. Then,  $\mathcal{F}$  is ordered by set inclusion in the sense that  $E$  is less than  $F$ , or  $E \leq F$  if and only if  $E \subset F$  ( $E, F \in \mathcal{F}$ ), so that each finite subset of  $\mathcal{F}$  has the least upper bound. Let  $E \in \mathcal{A}$ . If  $\mathcal{A}_E$  denotes the  $\sigma$ -algebra of all intersections of members of  $\mathcal{A}$  with  $E$  and  $\mu_E$  denotes the restriction of  $\mu$  to  $\mathcal{A}_E$ , then the triple  $(E, \mathcal{A}_E, \mu_E)$  is a positive measure space. For  $1 \leq p < \infty$ , let  $\mathcal{L}^p(E)$  be the vector space of measurable functions  $f$  defined on  $E$  for which  $\|f\|_{E,p} = (\int_E |f|^p d\mu)^{\frac{1}{p}} < \infty$  and let  $\mathcal{L}^\infty(E)$  be the vector space of measurable functions  $f$  defined on  $E$  for which  $\|f\|_{E,\infty} = \inf_N \sup_{w \in E \setminus N} |f(w)| < \infty$ , where  $N$  ranges over the null subsets of  $E$ . If  $\mathcal{N}_E$  denotes the set of null functions defined on  $E$  and  $[f]$  denotes the equivalence class of an  $f \in \mathcal{L}^p(E) \bmod \mathcal{N}_E$  ( $1 \leq p \leq \infty$ ), then  $[f] \mapsto \|f\|_{E,p}$  is a norm on the quotient space  $\mathcal{L}^p(E)/\mathcal{N}_E$ , which thus becomes a Banach space, denoted by  $L^p(E)$ . In particular, if  $\mu$  is the counting measure on  $\mathbb{N}$ , then we write  $l^1$  in place of  $L^1(\mathbb{N})$ . For each  $f \in L^p(\Omega)$ ,  $\|f\|_{\Omega,p}$  is called the  $L^p$ -norm of  $f$ , simply denoted by  $\|f\|_p$ . A measurable function  $f$  defined on  $\Omega$  is called essentially-bounded if  $\|f\|_\infty < \infty$ . Every element of  $L^p(E)$  is considered as a measurable function  $f$  defined on  $E$  with  $\|f\|_{E,p} < \infty$ , if no confusion will occur. For each  $E \in \mathcal{A}$ , the bilinear form  $(f, h) \mapsto \int_E fh d\mu$  on  $L^1(E) \times L^\infty(E)$  places  $L^1(E)$  and  $L^\infty(E)$  in duality. For  $E, F \in \mathcal{F}$  with  $E \leq F$ , let  $i_{EF}$  denote the mapping of  $L^1(F)$  onto  $L^1(E)$  that assigns to each  $f \in L^1(F)$  the restriction  $f|_E$  of  $f$  to  $E$ . Then, the canonical imbedding of  $L^\infty(E)$  into  $L^\infty(F)$  is the adjoint operator of  $i_{EF}$ , denoted by  $j_{FE}$ .

Let  $\mathcal{L}_{loc}^1(\Omega)$  be the vector space of measurable functions  $f$  defined on  $\Omega$  for which  $\|f\|_{E,1} < \infty$  for each  $E \in \mathcal{F}$  and let  $\mathcal{N}_{loc}$  be the vector subspace of  $\mathcal{L}_{loc}^1(\Omega)$  consisting of measurable functions  $f$  defined on  $\Omega$  for which  $\|f\|_{E,1} = 0$  for each  $E \in \mathcal{F}$ . If  $[f]$  denotes the equivalence class of an  $f \in \mathcal{L}_{loc}^1(\Omega) \bmod \mathcal{N}_{loc}$ , then  $[f] = [g]$  ( $f, g \in \mathcal{L}_{loc}^1(\Omega)$ ) means that for each  $E \in \mathcal{F}$ ,  $f(x) = g(x)$  almost everywhere on  $E$ . In particular, if  $\mu$  is  $\sigma$ -finite, then  $\mathcal{N}_{loc}$  equals the set  $\mathcal{N}_\Omega$  of null functions

defined on  $\Omega$  and hence for  $f, g \in \mathcal{L}_{loc}^1(\Omega)$ ,  $[f] = [g]$  if and only if  $f(x) = g(x)$  almost everywhere on  $\Omega$ . For each  $E \in \mathcal{F}$ ,  $[f] \mapsto \|f\|_{E,1}$  is a semi-norm on the quotient space  $\mathcal{L}_{loc}^1(\Omega)/\mathcal{N}_{loc}$ , which becomes a locally convex space, denoted by  $L_{loc}^1(\Omega)$ , under the separated locally convex topology  $\tau$  generated by the semi-norms  $[f] \mapsto \|f\|_{E,1}$  ( $E \in \mathcal{F}$ ). Every element of  $L_{loc}^1(\Omega)$  is considered as a measurable function  $f$  defined on  $\Omega$  for which  $\|f\|_{E,1} < \infty$  for each  $E \in \mathcal{F}$ , if no confusion will occur. If  $\mu$  is finite, then  $L_{loc}^1(\Omega)$  equals  $L^1(\Omega)$  and hence  $\tau$  is just the topology on  $L^1(\Omega)$  generated by the metric  $(f, g) \mapsto \|f - g\|_1$ .

In the sequel, we shall assume that the measure space  $(\Omega, \mathcal{A}, \mu)$  is  $\sigma$ -finite. The product space  $\mathcal{L}$  is the Cartesian product  $L = \prod_{E \in \mathcal{F}} L^1(E)$  of the family  $\{(L^1(E), \|\cdot\|_{E,1}) : E \in \mathcal{F}\}$  with its product topology. Then,  $L_{loc}^1(\Omega)$  is identified as a closed (and hence complete) subspace of  $\mathcal{L}$  by the isomorphism  $f \mapsto (f|_E)_{E \in \mathcal{F}}$  of  $L_{loc}^1(\Omega)$  into  $\mathcal{L}$ , where  $f|_E$  is the restriction of  $f$  to  $E$ . Let  $D = \bigoplus_{E \in \mathcal{F}} L^\infty(E)$  be the direct sum of the family  $\{L^\infty(E) : E \in \mathcal{F}\}$ . The vector spaces  $L$  and  $D$  are placed in duality by the bilinear form  $(f, g) \mapsto \sum_E \langle f_E, g_E \rangle$  on  $L \times D$ , where  $f = (f_E) \in L, g = (g_E) \in D$  and the sum is taken over at most a finite number of non-zero terms of  $g$ . Then, the topological dual of  $\mathcal{L}$  is  $D$  and the topological dual of  $L_{loc}^1(\Omega)$  is the quotient space  $D/(L_{loc}^1(\Omega))^\circ$ , which is algebraically isomorphic to the vector subspace  $L_{loc}^\infty(\Omega)$  of  $L^\infty(\Omega)$  consisting of measurable, essentially-bounded functions  $f$  defined on  $\Omega$  for which  $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$ .

**Proposition 1.**  $L_{loc}^1(\Omega)$  is a complete locally convex space. The topological dual of  $L_{loc}^1(\Omega)$  is algebraically isomorphic to  $L_{loc}^\infty(\Omega)$ .

We note that  $L_{loc}^1(\Omega)$  is identified as the reduced projective limit  $\varprojlim i_{EF} L^1(F)$  of the family  $\{(L^1(E), \|\cdot\|_{E,1}) : E \in \mathcal{F}\}$  with respect to the mappings  $i_{EF}$  ( $E, F \in \mathcal{F}$  and  $E \leq F$ ). If  $\mathcal{D} = \bigoplus_{E \in \mathcal{F}} L^\infty(E)$  denotes the locally convex direct sum of the family  $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$ , then the quotient space  $\mathcal{D}/(L_{loc}^1(\Omega))^\circ$  is the inductive limit  $\varinjlim j_{FE} L^\infty(E)$  of the family  $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$  with respect to the mappings  $j_{FE}$  ( $E, F \in \mathcal{F}$  and  $E \leq F$ ).

A subset  $A$  of  $L_{loc}^1(\Omega)$  is said to be locally uniformly integrable if for each  $E \in \mathcal{F}$ , the set  $\{f|_E : f \in A\}$  of the restrictions  $f|_E$  of the functions  $f$  in  $A$  to  $E$  is uniformly integrable in  $L^1(E)$ , that is, for each  $E \in \mathcal{F}$  and  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for each  $F \in \mathcal{A}$  with  $F \subset E$  and  $\mu(F) < \delta$ ,  $\sup_{f \in A} \int_F |f| d\mu < \epsilon$ . It follows from the theorem of Tychonoff that if  $A$  is a locally uniformly integrable, bounded subset of  $L_{loc}^1(\Omega)$ , then  $A$  is relatively weakly compact, since  $L_{loc}^1(\Omega)$  is a complete subspace of  $\mathcal{L}$ . The converse holds.

**Proposition 2.** A subset  $C$  of  $L_{loc}^1(\Omega)$  is relatively weakly compact if and only if  $C$  is bounded and locally uniformly integrable.

*Remark 1.* The arguments discussed so far is applicable for  $\sigma$ -compact topological spaces  $X$ . In this case, we choose as  $\mathcal{A}$  the  $\sigma$ -algebra of

Borel sets of  $X$  and as  $\mu$  a Borel measure on  $X$  such that  $\mu(K) < \infty$  for each compact subset  $K$  of  $X$ . For example, let  $X = \mathbb{R}$ , let  $\mu$  be Lebesgue measure on  $\mathbb{R}$ , let  $\mathcal{K}$  be the family of compact subsets of  $\mathbb{R}$  and let  $L^1_{loc}(\mathbb{R})$  be the space of Borel measurable functions  $f$  defined on  $\mathbb{R}$  for which  $\|f\|_{K,1} = \int_K |f| d\mu < \infty$  ( $K \in \mathcal{K}$ ), endowed with the separated locally convex topology generated by the semi-norms  $f \mapsto \|f\|_{K,1}$  ( $K \in \mathcal{K}$ ). Then,  $L^1_{loc}(\mathbb{R})$  contains the space  $C(\mathbb{R})$  of continuous (not necessarily bounded) functions defined on  $\mathbb{R}$ . If a subset  $\mathcal{C}$  of  $C(\mathbb{R})$  is uniformly bounded on the compact subsets of  $\mathbb{R}$ , that is,  $\sup_{f \in \mathcal{C}} \sup_{x \in K} |f(x)| < \infty$  ( $K \in \mathcal{K}$ ), then  $\mathcal{C}$  is relatively weakly compact in  $L^1_{loc}(\mathbb{R})$ .

We recall that whenever  $E$  is a metrizable locally convex space, then a subset  $C$  of  $E$  is weakly compact if and only if  $C$  is sequentially weakly compact.

**Proposition 3.** *A subset  $C$  of  $L^1_{loc}(\Omega)$  is weakly compact if and only if  $C$  is sequentially weakly compact.*

### 3. ON WEAK COMPACTNESS IN A SEPARATED LOCALLY CONVEX TOPOLOGY ON $L^1$

In this section,  $L^1(\Omega)$  shall be considered as a locally convex space under the separated locally convex topology  $\tilde{\tau}$  generated by the semi-norms  $f \mapsto \|f\|_{E,1}$  ( $E \in \mathcal{F}$ ), if  $L^1(\Omega)$  is not specified explicitly as a Banach space with the norm  $f \mapsto \|f\|_1$ , and we show a necessary and sufficient condition for a subset of  $L^1(\Omega)$  to be relatively weakly compact. It is clear that  $\tilde{\tau}$  is the relative topology of  $\tau$  on  $L^1_{loc}(\Omega)$  to  $L^1(\Omega)$ , since  $L^1(\Omega)$  is a subspace of  $L^1_{loc}(\Omega)$ . The topological dual of  $L^1(\Omega)$  is algebraically isomorphic to  $L^\infty_{loc}(\Omega)$ . The result concerning completeness of  $L^1(\Omega)$  follows immediately from the separation theorem.

**Proposition 4.** *The completion of  $(L^1(\Omega), \tilde{\tau})$  is  $L^1_{loc}(\Omega)$ .*

We showed a sufficient condition for a subset of  $L^1(\Omega)$  to be relatively weakly compact to obtain the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on  $L^1$ ; see [6].

**Proposition 5.** *Let  $C$  be a bounded subset of  $L^1(\Omega)$  relative to  $L^1$ -norm, that is,  $\sup_{f \in C} \|f\|_1 < \infty$ . Then,  $C$  is relatively weakly compact in  $(L^1(\Omega), \tilde{\tau})$  if and only if  $C$  is locally uniformly integrable.*

*Example 1.* Let  $\Omega = \mathbb{R}$ , let  $\mathcal{A}$  be the  $\sigma$ -algebra of Lebesgue measurable subsets of  $\mathbb{R}$ , let  $\mu$  be Lebesgue measure on  $\mathbb{R}$ , let  $\mathcal{F}$  be the family of Lebesgue measurable subsets of  $\mathbb{R}$  with finite measure. Let  $L^1(\mathbb{R})$  be endowed with the separated locally convex topology  $\tilde{\tau}$  generated by the semi-norms  $f \mapsto \|f\|_{E,1}$  ( $E \in \mathcal{F}$ ) and let  $f \in L^1(\mathbb{R})$ . For each  $y \in \mathbb{R}$ ,  $f_y$  is the translate of  $f$ , that is,  $f_y(x) = f(x - y)$  ( $x \in \mathbb{R}$ ). Then,  $\{f_y : y \in \mathbb{R}\}$  is relatively weakly compact in  $(L^1(\mathbb{R}), \tilde{\tau})$ . For example,

for  $f(x) = e^{-|x|}$  ( $x \in \mathbb{R}$ ),  $\{f_y : y \in \mathbb{R}\}$  is not relatively weakly compact in  $(L^1(\mathbb{R}), \|\cdot\|_1)$ , but relatively weak compact in  $(L^1(\mathbb{R}), \tilde{\tau})$ ; see also Remark 3.

*Example 2.* The closed unit ball in  $l^1$  is weakly compact in the topology of pointwise convergence, due to Fatou's lemma.

*Example 3.* Let  $\Omega, \mathcal{A}, \mu, \mathcal{F}$  and  $\tilde{\tau}$  be as in Example 1. Let  $f_n$  ( $n \in \mathbb{N}$ ) be a characteristic function on  $[n, 2n)$ . Then,  $\{f_n\}$  is not bounded relative to  $L^1$ -norm, but it converges to a null function in the topology  $\tilde{\tau}$  on  $L^1(\mathbb{R})$ . Thus, it is relatively compact and hence relatively weakly compact in  $(L^1(\mathbb{R}), \tilde{\tau})$ .

A subset  $C$  of  $L^1(\Omega)$  is said to be *locally bounded* if  $C$  is a bounded subset of  $L^1(\Omega)$ , that is, for each  $E \in \mathcal{F}$ ,  $\sup_{f \in C} \|f\|_{E,1} < \infty$ .

We show a necessary and sufficient condition for a subset of  $L^1(\Omega)$  to be relatively weakly compact in  $(L^1(\Omega), \tilde{\tau})$ .

**Theorem 1.** *A subset  $C$  of  $L^1(\Omega)$  is relatively weakly compact in  $(L^1(\Omega), \tilde{\tau})$  if and only if  $C$  is locally uniformly integrable, locally bounded and for each sequence  $\{f_n\}$  in  $C$ ,*

$$\sup_{E \in \mathcal{F}} \liminf_{n \rightarrow \infty} \left| \int_E f_n d\mu \right| < \infty.$$

#### 4. ON WEAK COMPACTNESS IN $L^1$

In the sequel,  $L^1(\Omega)$  shall be considered as a Banach space under the norm  $f \mapsto \|f\|_1$ . From Theorem 1, it is natural to ask a question of under which conditions every locally uniformly integrable, locally bounded subset of  $L^1(\Omega)$  is relatively weakly compact in  $(L^1(\Omega), \|\cdot\|_1)$ .

The following theorem is due to Grothendieck.

**Theorem 2.** *A subset  $C$  of a Banach space  $E$  is relatively weakly compact if and only if for each  $\epsilon > 0$ , there exists a weakly compact subset  $D$  of  $E$  such that  $C \subset D + B(\epsilon)$ .*

Motivated by his result, we introduce a notion of the type of uniform integrability to obtain a necessary and sufficient condition for a subset of  $L^1(\Omega)$  to be relatively weakly compact. We call a subset  $C$  of  $L^1(\Omega)$  *uniformly integrable at infinity* if for each  $\epsilon > 0$ , there exists an  $E \in \mathcal{F}$  such that

$$\sup_{f \in C} \int_{\Omega \setminus E} |f| d\mu < \epsilon \quad \left( \text{or } \limsup_{E \in \mathcal{F}} \sup_{f \in C} \int_{\Omega \setminus E} |f| d\mu = 0 \right).$$

**Theorem 3.** *Let  $C$  be a subset of  $L^1(\Omega)$ . Then, the following are equivalent:*

- (1)  $C$  is relatively weakly compact;

- (2) for each  $\epsilon > 0$ , there exists an  $E \in \mathcal{F}$  such that  $C_E = \{f|_E : f \in C\}$  is uniformly integrable, bounded in  $L^1(E)$  and  $C \subset C_E + B(\epsilon)$ ;
- (3)  $C$  is locally bounded, locally uniformly integrable and uniformly integrable at infinity;
- (4)  $C$  is bounded, uniformly integrable and uniformly integrable at infinity;
- (5)  $|C| = \{|f| : f \in C\}$  is relatively weakly compact, where  $|f|(x) = |f(x)|$  ( $x \in \Omega$ );
- (6)  $C$  is bounded and for each decreasing sequence  $\{E_n\}$  in  $\mathcal{A}$  with empty intersection,  $\int_{E_n} f d\mu$  converges to 0 uniformly in  $f \in C$ ;
- (7)  $C$  is bounded and there exists an  $f \in L^1(\Omega)$  such that for each  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for each  $E \in \mathcal{A}$  with  $\int_E |f| d\mu < \delta$ ,  $\sup_{g \in C} |\int_E g d\mu| < \epsilon$ .

*Remark 2.* The equivalence (1)  $\Leftrightarrow$  (6) is due to Dunford and Pettis, according to [3] and (1)  $\Leftrightarrow$  (7) is obtained as in Bartle, Dunford and Schwartz [1]. The latter implies that Theorem 3, Theorem 4 and Corollary 1 hold without the assumption that  $\mu$  is  $\sigma$ -finite, since every function in a weakly compact subset of  $L^1(\Omega)$  vanishes on the complement of a  $\sigma$ -finite set.

*Remark 3.* Let  $\Omega, \mathcal{A}$  and  $\mu$  be as in Example 1. Then, the subset of  $L^1(\mathbb{R})$  consisting of the translates of  $f(x) = e^{-|x|}$  ( $x \in \mathbb{R}$ ) is not uniformly integrable at infinity and hence is not weakly compact in  $L^1(\mathbb{R})$ .

**Corollary 1.** *Every order interval in  $L^1(\Omega)$  is weakly compact, where an order interval is a subset of the form  $\{h \in L^1(\Omega) : f(x) \leq h(x) \leq g(x) \text{ almost everywhere on } \Omega\}$  ( $f, g \in L^1(\Omega)$ ).*

The following theorem is due to Theorem 3 and the convergence theorem of Vitali.

**Theorem 4.** *Let  $C$  be a weakly compact subset of  $L^1(\Omega)$ , let  $\{f_n\}$  be a sequence in  $C$  and let  $f \in C$ . If  $f_n(x)$  converges to  $f(x)$  almost everywhere on  $\Omega$ , then  $\|f_n - f\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .*

**Corollary 2.** *Every weakly convergent sequence in  $l^1$  is strongly convergent.*

## 5. ON STRONG COMPACTNESS IN $L^1$

Note that in the sequel,  $L^1(\Omega)$  shall be considered as a Banach space under the norm  $f \mapsto \|f\|_1$ . As in the arguments in the previous sections,  $L^\infty(\Omega)$  is considered as a locally convex space under the separated locally convex topology  $\hat{\tau}$  generated by the semi-norms  $f \mapsto \|f\|_{E,1}$  ( $E \in \mathcal{F}$ ). It is clear that  $\hat{\tau}$  is the relative topology of  $\tau$  on  $L^1_{loc}(\Omega)$  to  $L^\infty(\Omega)$ , since  $L^\infty(\Omega)$  is a subspace of  $L^1_{loc}(\Omega)$ . The topological dual

of  $(L^\infty(\Omega), \hat{\tau})$  is algebraically isomorphic to  $L_{loc}^\infty(\Omega)$ . The result concerning completeness of  $(L^\infty(\Omega), \hat{\tau})$  also follows immediately from the separation theorem.

**Proposition 6.** *The completion of  $(L^\infty(\Omega), \hat{\tau})$  is  $L_{loc}^1(\Omega)$ .*

The weak topology  $\sigma(L^\infty(\Omega), L^1(\Omega))$ , simply denoted by  $\sigma(L^\infty, L^1)$ , is finer than the weak topology of  $L^\infty(\Omega)$  associated with  $\hat{\tau}$ , from which we directly deduce the following result concerning sequential compactness in  $\sigma(L^\infty, L^1)$  on  $L^\infty(\Omega)$ .

**Proposition 7.** *The closed unit ball in  $L^\infty(\Omega)$  is sequentially compact relative to the weak topology  $\sigma(L^\infty, L^1)$ .*

*Remark 4.* If  $E$  is a reflexive or smooth Banach space, then every closed unit ball in  $E'$  is sequentially compact relative to the weak topology  $\sigma(E', E)$ ; see [2] for more details.

A subset  $C$  of a Banach space  $E$  is said to be *limited* if for each sequence  $\{x'_n\}$  in  $E'$  converging to 0 in the weak topology  $\sigma(E', E)$ ,  $\lim_{n \rightarrow \infty} |\langle x, x'_n \rangle|$  converges to 0 uniformly in  $x \in C$ .

Using similar arguments to [9], we obtain a characterization of strong compactness in Banach spaces  $E$  for which the closed unit ball in  $E'$  is sequentially compact relative to the weak topology  $\sigma(E', E)$ .

**Proposition 8.** *Let  $E$  be a Banach space. Whenever the closed unit ball in  $E'$  is sequentially compact relative to the weak topology  $\sigma(E', E)$ , a subset  $C$  of  $E$  is relatively compact if and only if  $C$  is bounded and limited.*

*Remark 5.* According to [2], Proposition 8 is due to Gelfand.

**Corollary 3.** *Whenever  $E$  is a reflexive or smooth Banach space, a subset  $C$  of  $E$  is relatively compact if and only if  $C$  is bounded and limited.*

The following theorem is due to Proposition 7 and Proposition 8.

**Theorem 5.** *A subset  $C$  of  $L^1(\Omega)$  is relatively compact if and only if  $C$  is bounded and limited.*

*Remark 6.* It is clear that Theorem 5 holds without the assumption that  $\mu$  is  $\sigma$ -finite.

## 6. MISCELLANEOUS APPLICATIONS

In this section, we apply the results about weak and strong compactness in  $L^1(\Omega)$  to obtain some characterizations of (weak) almost periodicity for linear and non-linear operators in  $L^1(\Omega)$ .

Let  $T$  be a linear contraction on  $L^1(\Omega)$ , that is,  $T$  is a linear operator on  $L^1(\Omega)$  such that  $\|Tf\|_1 \leq \|f\|_1$  ( $f \in L^1(\Omega)$ ). In addition, if  $\|Tf\|_\infty \leq \|f\|_\infty$  ( $f \in L^1(\Omega) \cap L^\infty(\Omega)$ ), then  $T$  is said to be a



Dunford-Schwartz operator on  $L^1(\Omega)$ . If for each  $f \in L^1(\Omega)$ , the orbit  $\{T^n f : n = 0, 1, 2, \dots\}$  of  $f$  under  $T$  is relatively (weakly) compact, then  $T$  is said to be (weakly) almost periodic.

**Proposition 9.** *Let  $T$  be a Dunford-Schwartz operator on  $L^1(\Omega)$ . Then,  $T$  is weakly almost periodic if and only if for each  $f \in L^1(\Omega)$ , the orbit of  $f$  under  $T$  is uniformly integrable at infinity.*

**Proposition 10.** *Let  $T$  be a linear contraction on  $L^1(\Omega)$ . Then,  $T$  is almost periodic if and only if for each  $f \in L^1(\Omega)$ , the orbit of  $f$  under  $T$  is limited.*

Let  $C$  be a closed convex subset of  $L^1(\Omega)$  and let  $T$  be a nonexpansive operator on  $C$ , that is,  $T$  is a mapping of  $C$  into itself such that  $\|Tf - Tg\|_1 \leq \|f - g\|_1$  ( $f, g \in C$ ). Then,  $T$  is said to be *almost periodic* if for each  $f \in C$ , the orbit  $\{T^n f : n = 0, 1, 2, \dots\}$  of  $f$  under  $T$  is relatively compact. It is known that if a nonexpansive operator  $T$  on  $C$  is almost periodic, then  $T$  has the mean values on  $C$ ; see also [4] for more details.

**Proposition 11.** *Let  $C$  be a closed convex subset of  $L^1(\Omega)$  and let  $T$  be a nonexpansive operator on  $C$ . Whenever  $T$  has a fixed point in  $C$ ,  $T$  is almost periodic if and only if for each  $f \in C$ , the orbit of  $f$  under  $T$  is limited.*

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